Algorithm and Demonstration in the Sixteenth-Century *Ars Magna*

Renaissance mathematicians in sixteenth-century Italy and France sought to humanize algorithmic mathematics. By then, algorithm had been well-enough known to a nascent Europe for some time.[1] Manuals like Fibonacci's *Liber Abaci* disseminated to merchants and others the practice of doing arithmetical calculations using Arabic numerals.[2] In the sixteenth century new works were written to teach to students the rules of numeral calculation, which was sometimes called "algorism" in English.[3] In these contexts algorithm was a calculation practice, and so it remained in the textbooks on algebra and the *ars magna*, or great art, that I discuss here. But in these works it was not just a practice. Incorporating algorithm into established educational institutions meant assimilating it to mathematical learning as already understood. This meant theories—of education, of demonstration, of metaphysics.

When mathematical authors sought to naturalize algorithmic practice within humanist learning, they did so by associating it with the seven liberal arts that together constituted preliminary university education in Renaissance Europe. At times, they leaned heavily on the identification of algebra with arithmetic. But there were other options. As has been recognized, university pedagogy played a central role in the diffusion of humanism in France and the Low Countries. Reforming mathematics pedagogy was a central plank of humanist education at the turn

[1] On the role of histories of algebra in the formation of mathematical Europe, see Jens Høyrup, "The Formation of a Myth: Greek Mathematics — Our Mathematics," and Giovanna Cifoletti, "The Creation of the History of Algebra in the Sixteenth Century," both in *L'Europe mathématique: Histoires, Mythes, Identités*, ed. Catherine Goldstein, Jeremy Gray, and Jim Ritter (Paris: Éditions de la Maison des sciences de l'homme, 1996), 103-119 and 123-142.
[2] See, *inter alia*, Karen Hunger Parshall, "A Plurality of Algebras, 1200-1600: Algebraic Europe from Fibonacci to Clavius," *BSHM Bulletin: Journal of the British Society for the History of Mathematics* 32.1 (2017): 2-16 and, on the French story, Warren van Egmond, "How Algebra Came to France," *Mathematics from Manuscript to Print 1300-1600*, ed. Cynthia Hay (Oxford: Clarendon Press, 1988), 127-144.
[3] In this volume Michael Barany discusses a prominent example: Robert Recorde's 1543 *The Ground of Artes*. For a brief discussion of algorism and arithmetic, see Angela Axworthy, *Le Mathématicien renaissant et son savoir. Le statut des mathématiques selon Oronce Fine* (Paris: Classiques Garnier, 2016), 288-9, and literature cited there.

of the sixteenth century.[4] A series of studies by Giovanna Cifoletti forms the background to my argument here; she shows that French humanists took advantage of this institution by associating framing algebra as a perfected form of dialectic and thus as a replacement for the *trivium* of logic, rhetoric, and grammar no less than as a replacement for the *quadrivium*. This new kind of learning was called *mathesis*.[5] And in yet another interpretation of *mathesis*, namely a metaphysical one consistent with Aristotelian theories of science, algebra seemed to some writers to unify arithmetic and geometry by identifying a subject common to both.[6] All of these approaches involved efforts to change algorithmic practice no less than new justifications for it.

I focus here on a narrow slice of these efforts, namely the integration of algorithms into Euclidean geometric demonstration. Algebra was the subject in which this integration most conspicuously took place.[7] In book II of the *Elements* Euclid used geometric techniques to attack the relationship between sides and squares; in later books he broached the subjects of proportions between numbers, in book V, and between figures, in book VII. The study of proportions was central to medieval and early modern mathematics, and Greek mathematics provided ample resources for addressing it.[8] Meanwhile, just this material was addressed efficaciously by the algorithms of the great art. How this complementarity in disciplinary subject matter was

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[4] Anthony Grafton and Lisa Jardine, *From Humanism to the Humanities: Education and the Liberal Arts in Fifteenth- and Sixteenth-Century Europe* (Cambridge, MA: Harvard University Press, 1986); on mathematics, see Ann Moyer, "Reading Boethius on Proportion: Renaissance Editions, Epitomes, and Virions of the Arithmetic and Music," in Rommevaux, Vendrix, and Zara, eds. (2012), 51-68, and Richard J. Oosterhoff, *Making Mathematical Culture: University and Print in the Circle of Lefèvre d'Étaples* (Oxford: Oxford University Press, 2018).

[5] Giovanna Cifoletti, "La question de l'algèbre: Mathématiques et rhétorique des hommes de droit dans la France du 16e siècle," *Annales. Histoire, Sciences Sociales* 50.6 (Nov.-Dec. 1995): 1385-1416; "L'utile de l'entendement et l'utile de l'action: discussion sur la utilité des mathématiques au xvie siècle," *Revue de synthèse* 4.2-4 (Apr.-Dec. 2001): 503-20.

[6] Katherine Neal, *From Discrete to Continuous: The Broadening of Number Concepts in Early Modern England* (Dordrecht: Springer, 2002); David Rabouin, *Mathesis Universalis: L'idée de mathématique universelle d'Aristote à Descartes* (Paris: Presses universitaires de France, 2009).

[7] On sixteenth-century algebra, the essential starting point is Sabine Rommevaux, Maryvonne Spiesser, and Maria Rosa Massa Esteve, eds., *Pluralité de l'algèbre à la Renaissance* (Paris: Honoré Champion, 2012).

[8] See contributions in Sabine Rommevaux, Philippe Vendrix, and Vasco Zara, eds., *Proportions. Science, Musique, Peinture & Architecture* (Turnhout: Brepols Publishers, 2012).

recognized and accommodated theoretically has been much discussed.[9] It suffices here to note that the integration of algorithms into demonstrations implicated elements from both interpretations of algebra as a form of *mathesis*. Not the only discipline of university mathematics in the sixteenth century, Euclidean geometry was nevertheless then the highest status both cognitively and metaphysically.[10] Accordingly, the use of arithmetical and algorithmic demonstrations had potential ramifications both for cognitive dimensions of persuasion (hence for Renaissance dialectic) and for the relationship between arithmetic and geometry (hence for the ontology of mathematics).

My subject here is how mathematical authors combined, within particular demonstrations, the genre of demonstration and the algorithmic rule. My interest is in the relationship between algorithmic procedure and persuasion: can following an algorithm lead to some kind of knowledge about the truth of a mathematical statement or belief about the reliability of a rule? Or must this knowledge or persuasion come from elsewhere? Algebra books from Italy, the Netherlands, and France—by Girolamo Cardano, Simon Stevin, and Guillaume Gosselin—illustrate instances of this relationship during the half-century during which algorithms took center-stage in the study of proportions.

I begin by looking at Cardano's mid-century *Book of the Great Art or of Algebraic Rules*, which uses diagram-based geometry to demonstrate rules for solving cubic equations. In Cardano's

[9] See Rabouin (2009) and, earlier, Jacob Klein, *Greek Mathematical Thought and the Origin of Algebra*, trans. Eva Brann (Mineola, New York: Dover Publications, Inc., 1992 [reissue of 1968 edition]). The issue is broached in a new way, from a pedagogical perspective, in Oosterhoff (2018).

[10] Debates about the certitude of mathematics, even when they question the status of Euclidean demonstration vis-à-vis Aristotelian syllogism, obviously accept that demonstration (rather than, say, algorism) is the relevant point of comparison to logic. On such debates, see Nicholas Jardine, "The Epistemology of the Sciences," in *The Cambridge History of Renaissance Philosophy*, eds. Charles, Schmitt, Quentin Skinner, Eckhard Kessler, and Jill Kraye (Cambridge: Cambridge University Press, 1988), 685-711; Paolo Mancosu, "Aristotelian Logic and Euclidean Mathematics: Seventeenth-Century Developments of the *Quaestio de Certitudine Mathematicarum*," *Studies in the History and Philosophy of Science* 23.2 (1992): 241-265.

text the reliability of rules is a consequence of traditional geometric demonstrations, and each rule reflects its demonstration and so its justification. In contrast to this approach, algorithms play a central role in the demonstrations developed several decades later by both Stevin and Gosselin. Both authors integrate algorithms into Euclidean demonstration by accommodating the standard form of the demonstration to the cognitive, and paper, practices made available by the algorithm. My goal is to identify some possibilities for mathematical cognition that were implicit in the incorporation of algorithms into the Euclidean demonstration. This investigation is important precisely because in our contemporary world the use of algorithms is often presented as somehow anti-cognitive, as requiring humans to behave like machines. Has this always been true? Has it ever been true? The relationship between putatively automatic algorithms and eminently intellectual demonstration is one place to look.

Form as a guide to procedure: Cardano's diagrams

In his *Great Art* Cardano founded an algorithmic approach to cubic equations on a geometric basis. Consistent with the kind of claim typically found in algorithmic texts, Cardano formulated "rules" for the solution of cubic equations.[11] He justified these rules using typical geometric demonstrations that he carried out on a figure. Recollections from high-school algebra might suggest to some readers that Cardano used equations that could be manipulated. On the contrary, the stable forms of Cardano's equations served a different purpose. To wit, where

[11] Girolamo Cardano, *Artis magnae, sive de regulis algebraicis, liber unus* (Nuremberg: Johannes Petreius, 1545); see also the recent critical edition, *Artis magnae, sive, De regulis algebraicis liber unus*, ed. Massimo Tamborini (Milan: FrancoAngeli, 2011), and in English translation, *The Rules of Algebra*, trans. T. Richard Witmer (Mineola, New York: Dover Publications, Inc., 1992 [reissue of 1968 edition]). For a discussion of the overall work, see Jacqueline Stedall, *From Cardano's* Great Art *to Lagrange's* Reflections*: Filling a Gap in the History of Algebra* (Zürich: European Mathematics Society, 2011), 3-17. On Cardano's career, see Anthony Grafton, *Cardano's Cosmos: The Worlds and Works of a Renaissance Astrologer* (Cambridge: Harvard University Press, 1999).

geometrical demonstrations justified rules that could be applied independently of the diagram, the forms of the equations served to demarcate the generality of their applicability.

The form of each equation both anchors the generality of Cardano's rule and marks out its limits. A modern reader might expect a general form of the equation and a single, closed-form procedure for its solution, such as $ax^2+bx+c = 0$ and the well-known quadratic formula taught in high school, $x = -b \pm \frac{\sqrt{b^2-4ac}}{2a}$. Instead, Cardano proposed multiple rules following differences in the equations' forms, differences such as that between $x^3 + bx = c$ and $x^3 = bx + c$ (note the shifted equals sign).[12] Each case—or in Cardano's words, "head" [*capitulum*]—had its own "mode" [*modus*] of the demonstration and its own rule.[13] For Cardano, the form facilitated procedure in two interrelated ways. First, it identified which rule to carry out. Second, it shaped the question so as to provide material for those procedures, much as the general equation ax^2+bx+c fixes values for a, b, and c that are used in the quadratic formula.

Beyond its use in marking out generality, the form of each particular case also participated in the justification that Cardano gave for each rule. Cardano connected each form to a diagrammatic representation of a cubic equation. Cardano used each diagram in order to demonstrate the corresponding rule. In the demonstrations, the diagrams kept track of the various parts of the form, namely the cubes, squares, sides, and number. By keeping track, they also facilitated calculations. By anchoring his rules in diagrams, Cardano claimed to give demonstrations that could produce "belief" [*fides*].[14] Let's look at an example.

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[12] Cardano distinguished different cases where we would see just one because he only accepted positive values for the root. Thus the equation $ax^2+bx+c = 0$ would give rise to multiple cases depending on whether a, b, and c are positive or negative.
[13] Cardano (1545), cap. 6, fol. 14v and cap. 11, fol. 29v = Cardano (1992), 48 and 96. These two heads are discussed in chapters 11 and 12 of Cardano's text. For $(b/3)^3 > (c/2)^2$ the latter mode is irreducible in real numbers, and Cardano restricted his rule to the alternative case. See Cardano (1545), fol. 31r = Cardano (1992), 103; and for discussion, see Stedall (2011), 10.
[14] Cardano (1545), cap. 1, fol. 6v = Cardano (1992), 20.

The sixth chapter introduced a diagram that would figure centrally in the rest of the book as well as a key concept: "substitution."[15] The diagram (fig. 1) expressed "three greatly useful substitutions [*supposita*]" that enabled his comprehensive treatment of all cases.

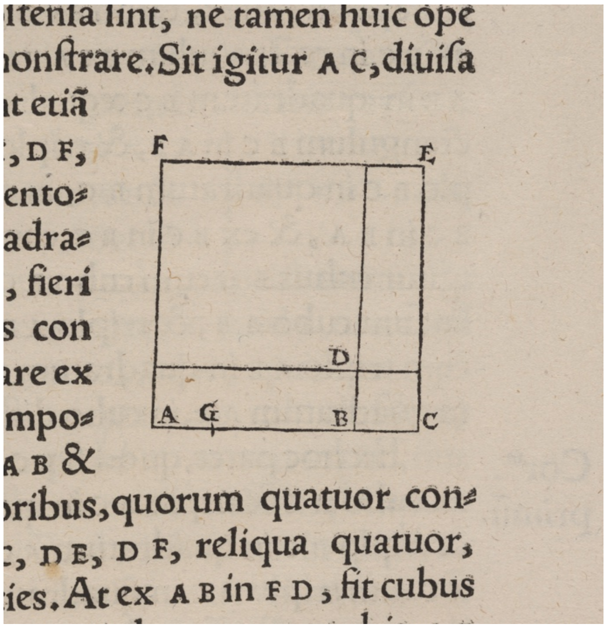

Figure 1. Erect a cube on the side AC. Cardano, Artis Magnae...liber (1545), cap. VI, fol. 16r.

Cardano elevated a cube on the square AC and collected its parts: "the cube *AE* will consist of eight bodies, four of which are made from line *AB* times the surfaces *DA*, *DC*, *DE*, and *DF*, and the remaining four from the line *BC* times the same four surfaces." Of these bodies, two are cubes (*AB* × *DF* and *BC* × *DC*) and six are parallelepipeds, of two different kinds (*AB* × *DA*, *AB* × *DE*, and *BC* × *DF* are equal; *AB* × *DC*, *BC* × *DA*, and *BC* × *DE* are equal). The cube imagined to rise upon the diagram arranged all the parts of the cube of a general binomial: if the side AC of a square is cut at point B, then $AC^3 = (AB + BC)^3 = AB^3 + 3AB^2 \times BC + 3AB \times BC^2 + BC^3$. This substitution of eight parts (two cubes and six parallelepipeds) for AC^3 furnished Cardano with a

[15] All the cases where none of the monomials has a 0 value (chapters 17-23) use it, and several of the earlier cases (chapters 11-16) use variations on it.

fixed relationship of volumes expressed in terms of the cubes, squares, and sides of two values, AB and BC.

Cardano used the substitution to organize the argument for each case. In order to do so, he had to assign appropriate values to the sides of the diagram. Take the case where "the cube is equal to things and a number" ($x^3 = bx + c$). In typical Euclidean style, Cardano constructed the solution (see again fig. 1) before he demonstrated it: "let DC and DF be two cubes whose sides AB and BC, when they are extended, produce by multiplication the third part of the number of things," namely the coefficient b of bx. Further, the sum of the cubes is equal to the number c. Then "I say that AC is the value of the sought number," x. Cardano's construction was fitting: by making AB × BC = $b/3$, Cardano obtained the equation 3 AB × BC = a and so the useful equation AC × (3 AB × BC) = ax. Since AC = AB + BC, the product AC × (3 AB × BC) makes all six parallelepipeds in the original cube. (Cardano advanced a subsidiary proof of this claim.) Recall that the sum of the cubes CD and DF is c. By appeal to the basic substitution, since the cube AC is composed by the six parallelepipeds and the two cubes CD and DF, it is equal to AC × (3 AB × BC) plus CD + DF, thus to $bx + c$, and thus to the cube of x.[16] By assigning appropriate values, such as AB × BC = $b/3$, Cardano could embed certain relationships in the diagram and then exploit them through substitutions. Substitutions in Cardano's great art were not algebraic procedures to be used in solving a particular problem, but ways of relating rules to their geometric justifications.

Cardano recalled that he turned to the great art and its use of unknown quantities because, after first learning the solution for one case of the cubic, he judged that expressing "known questions in terms of unknown positions [*ignotas positiones*]" would allow him to pursue "general

[16] Cardano (1545), cap. 12, fol. 31r = Cardano (1992), 102. Later demonstrations drew on this one: see, e.g., cap. 19, fol. 41r.

things from one question." He turned to geometry because he thought it could serve as a "royal road for coming to every case."[17] And it was: the substitutions embedded in the diagram allowed Cardano to tackle each case. The relationship between powers of the unknown guided Cardano in setting up each diagram, and the prepared diagram was in turn central to Cardano's demonstration of each rule. Subsequently, the form of each case both identified when a given rule should be used and provided material—like the coefficients *b* and *c* in the quadratic equation—for its use. Algorithmic automaticity and intellectual demonstration are interrelated, but side by side.

The role of algorithms in demonstrations

Reflecting their efforts to integrate algorithmic mathematics into the liberal arts, sixteenth-century writers about algorithms drew on two different organizational styles for presenting their mathematics: a style of Greek origin composed of theorems and demonstrations, typically used for university geometry, and a vernacular style composed of rules and examples, typically used for arithmetic.[18] As we saw, Cardano used the geometric style to justify arithmetical rules that could be applied independently of the diagram. Other writers integrated the two styles differently, and in ways that pushed against the division of intellectual labor that Cardano's organization suggested: a division between justification as the preserve of geometry and practice as the preserve of algorithms. This section looks at how mathematical writers following Cardano integrated

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[17] Cardano (1545), cap. vi, fol. 16r = Cardano (1992), 52. On reducibility and generality in Cardano's mathematics, see Stedall (2011), 11. Cardano identifies Niccolò Tartaglia in this context; on Tartaglia's later reception in by the mathematicians I discuss below, see Giovanna Cifoletti, "Mathematics and Rhetoric: Jacques Peletier, Guillaume Gosselin, and the Making of the French Algebraic Tradition," Princeton University PhD thesis, 1992, ch. 2.

[18] On commercial arithmetic, see the well-known studies by Natalie Zemon Davis: "Mathematicians in Sixteenth-Century French Academies: Some Further Evidence," *Renaissance News* XI (1958): 3-10 and "Sixteenth-Century French Arithmetics on the Business Life," *Journal of the History of Ideas* 21.1 (1960): 18-48; on the incorporation of arithmetic at the universities, see the important article by Jean-Claude Margolin, "L'Enseignement des mathématiques en France (1540-70): Charles de Bovelles, Fine, Peletier, Ramus," in *French Renaissance Studies 1540-70: Humanism and the Encyclopedia*, ed. Peter Sharratt (Edinburgh: Edinburgh University Press, 1976): 109-155.

algorithms into the Euclidean demonstrative scheme and thus used algorithms to produce knowledge or belief.

In mid-century arithmetical works, geometry often played a justificatory role. Both Jacques Peletier and Pierre Forcadel, who wrote French-language texts on arithmetic and algebra in the 1540s and 1550s, appealed to propositions from Euclid in order to justify (Peletier) or explain (Forcadel) the direct and inverse rules of three.[19] As Cifoletti has argued, French humanists took recourse to the language of theory and practice in order to explain their appeals to geometry. In terms that Guillaume Gosselin used in his 1583 *Renewed lecture on the way of teaching and learning mathematics*, the knowing part of arithmetic makes laws and puts together rules, while the active part uses those rules for action.[20] Arithmetical writers appealed to geometry for its theorems and put them to use in solving arithmetical or algebraic problems.[21]

Today's reader might suppose that geometry's justificatory role derived from the generality of its theorems, as distinguished from the particularity of any application of a rule. But in arithmetical writings, geometrical demonstrations were frequently particular. Cardano demonstrated several of his rules by assigning definite values to the coefficients of the unknown: his demonstration for "the cube and square equal to the number" began "Let $AB^3 + 6AB^2 = 100$" rather than with indefinite coefficients that we might expect and that would make his demonstration general.[22] When Forcadel appealed to geometry to supply "the understanding, perfection, and true intelligence of the rules of three," he instructed the reader to "impress in

[19] Jacques Peletier, *L'Aritmetique…departie in quatre Livres* (Poitiers: [Jean de Marnef], 1549); Pierre Forcadel, *L'Arithmeticque* (Paris: Guillaume Cavellat, 1557).
[20] Cifoletti (1995), 1392-3 and 1412; more generally, see Cifoletti (2001), esp. 514. For Gosselin's text, see the critical edition in Guillaume Gosselin, *De arte magna libri IV Traité d'algèbre suivi de Praelectio/Leçon sur la mathématique*, ed. and trans. Odile Le Guillou-Kouteynikoff (Paris: Les Belles Lettres, 2016), 406-483, at 422-425.
[21] On the use of geometric theorems, see also Jacques Peletier, *L'Algèbre…departie an deus Livres* (Lyon: Jean de Tournes, 1554), 113.
[22] Cardano (1545), cap. xiv, fol. 33r = Cardano (1992), 110.

yourself the idea of the forty-fourth proposition of the first book of Euclid, and propose yourself these three numbers, 2, 3, and 6" through which the fourth proportional might be found.[23] Forcadel apparently believed that working through the particular example would bring understanding more easily than recurring to the generality of Euclid's demonstration.

But what if Euclid's discussion wasn't general either? In the mid-seventeenth century the English mathematician John Wallis advanced just this interpretation of Greek geometric demonstration. A traditional scheme perhaps originating with the ancient philosopher Proclus divided Euclid's theorems into six parts: where the *protasis* (enunciation) and *superasma* (conclusion) stated a theorem in general terms, the four intervening steps stated and proved the same theorem by referring to a particular diagram using lettered objects.[24] In Wallis' view, the relationship between general result and particular demonstration was established by induction, by extending the results of one or several cases to all of them. Wallis summarized Euclid's construction of an equilateral triangle, the same example that Proclus had used to discuss the scheme: "He sets forth one right line AB, and demonstrates the triangle constructed on it to be equilateral… by the same method, on any other right line, a triangle can be constructed in this way and also demonstrated, therefore, in any one you like." Although Euclid performed the construction on just one line, it could be repeated on "any other." According to Wallis, the force of the demonstration rested in just this repeatability: "The force of this or of almost any other demonstration is not considered to lie anywhere else than in the supposition, that no case would

[23] Forcadel (1557), livre I, fol. 74r. For discussion of Forcadel's use of rules, see François Loget, "L'algèbre en France au XVIe siècle," *Pluralité de l'algèbre à la Renaissance*, ed. Rommevaux, Spiesser, and Massa Esteve (2012): 69-101.
[24] Proclus, *In Primum Euclidis Elementorum Librum Commentarii*, 203 ff.; *A Commentary on the First Book of Euclid's Elements*, trans. Glenn Morrow (Princeton: Princeton University Press, 1970), 159-162. For discussion of the scheme, see Reviel Netz, "Proclus' Division of the Mathematical Proposition into Parts: How and Why Was it Formulated?" *The Classical Quarterly* 49.1 (1999): 282-303.

be able to be pressed to the contrary to which the demonstration set forth could not be applied."[25] Wallis' argument was an interested one: he was hoping to legitimate his own use of mathematical induction in the 1655 *Arithmetica Infinitorum* by portraying Euclidean method as also based on induction. Nevertheless, his discussion of Euclid's very first proposition sheds light on the sixteenth-century practice of making demonstrations using single examples.

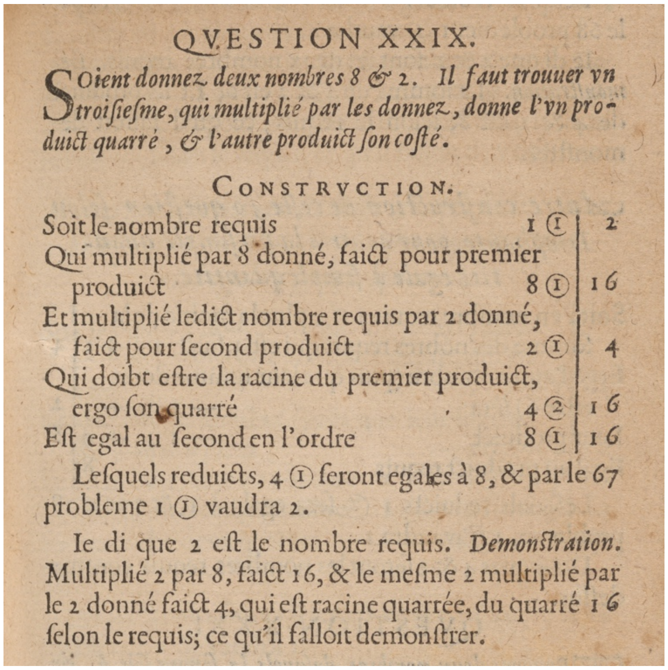
Sixteenth-century algebraic writers adapted the Euclidean scheme to arithmetic by placing worked algorithms in the place of geometric constructions. A rediscovered ancient work, Diophantus of Alexandria's *Arithmetica*, encouraged these authors in conceiving of a demonstrative arithmetic on the model of Euclidean geometry.[26] For Diophantus himself seemed to employ a version of the distinction between general enunciation and particular demonstration, setting out general algebraic results (like those demonstrated by Cardano) and then showing them by means of a particular example. The 1575 Latin translation by the German humanist Wilhelm Holtzman (usually called Xylander) underscored the parallel between geometric constructions and algorithms by transforming each problem into a "canon" or rule. "Since the arts master [*artifex*] is able to construct infinite canons even concerning things of great moment from algebraic operations," he wrote, "we also make a general canon from what has been demonstrated." Like a Euclidean conclusion, Xylander expressed the canon generally and at the end of the demonstration

[25] See John Wallis' letter to Christian Huygens of 12/22 August 1656, in John Wallis, *Correspondence*, ed. Philip Beeley and Christoph Scriba, 4 vol. (Oxford: Oxford University Press, 2003-), vol. I, 195: "Deinde exponit unam rectam *AB*, et super hanc constructum triangulum demonstrat esse aequilaterum: Tum subsumendum relinquit, (quod nempe quilibet supplere debet,) Atque eadem methodo, super quamvis aliam rectam ita construetur triangulum, et pariter demonstrabitur. Ergo, In quavis &c. Nec quidem aliter constat vis istius aliusve fere demonstrationis, quam ex suppositione, quod nullus casus in contrarium urgeri possit cui non applicabitur exposita demonstratio." Cf. John Wallis, *Due Correction for Mr Hobbes* (Oxford: Leonard Lichfield, 1656), 42.
[26] On the reception of Diophantus in the Renaissance, see Joann Stephanie Morse, "The Reception of Diophantus' 'Arithmetic' in the Renaissance," Princeton University PhD thesis, 1981; Ad Meskens, *Travelling Mathematics – The Fate of Diophantos' Arithmetic* (Basel: Springer, 2010).

(even after his own scholium).[27] Apparently sixteenth-century scholars saw little to distinguish geometric constructions made out of particular lines from algorithms using particular numbers, since they fulfilled similar roles in the demonstrative scheme.[28]

The assimilation of algorithms to geometric constructions is even clearer in Simon Stevin's 1585 *L'arithmetique* and accompanying edition of the first four books of Diophantus' *Arithmetic*.[29] Well-known as an engineer and pioneer in mechanics, Stevin was a polymath whose reflections on number and language reflected his classical learning.[30] His edition of Diophantus distinguished between general theorems and particular problems or, here, "questions."[31] In framing his theorems, Stevin employed a version of the traditional Euclidean scheme, whose parts he marked out using headings: a general statement and conclusion bracketed the "explanation of the given" and "of the sought" and the particular demonstration based on them. In his theorems, Stevin left out the "construction" of each demonstration. But Stevin's questions made explicit his understanding that a sequence of arithmetical operations was a sort of "construction," akin to Euclid's creation of the diagram (see fig. 2).

[27] See Diophantus of Alexandria, *Rerum arithmeticarum libri sex*, ed. Wilhelm Xylander (Basel: Eusebius and Nicolas Episcopius, 1575), 9. For Xylander's organization of Diophantus' problems, see *passim*. For a critical edition, see Diophantus of Alexandria, *Opera omnia*…, ed. Paul Tannery, 2 vols. (Leipzig: B. G. Teubner, 1893-95), but note well that Tannery's edition likewise exaggerates Diophantus' resemblance to Euclid: for discussion, see Reviel Netz, "Reasoning and symbolism in Diophantus: preliminary observations," in *The History of Mathematical Proof in Ancient Traditions*, ed. Karine Chemla (Cambridge: Cambridge University Press, 2012), 327-361.
[28] To what degree early modern geometric diagrams were understood as undetermined (in magnitude and position) or particular is beyond the scope this chapter; for discussion, see Vincenzo De Risi, ed., *Mathematizing Space: The Objects of Geometry from Antiquity to the Early Modern Age* (Cham et al.: Springer/Birkhäuser, 2015).
[29] Simon Stevin, *L'Arithmetique de Simon Stevin de Bruges: Contenant les computations des nombres Arithmetiques ou vulgaires: Aussi l'Algebre, avec les equations de cinc quantitez. Ensemble les quatre premiers livres d'ALgebre de Diophante d'Alexandrie, maintenant premierement traduicts en François* (Leiden: Christophe Plantin, 1585).
[30] On Stevin's life and career, see E. J. Dijksterhuis, *Simon Stevin: Science in the Netherlands around 1600*, trans. C. Dikshoorn (The Hague: Martinus Nijhoff, 1970)
[31] Stevin (1585), 431 ff., *passim*.



QVESTION XXIX.

Soient donnez deux nombres 8 & 2. Il faut trouuer vn troiſieſme, qui multiplié par les donnez, donne l'vn produict quarré, & l'autre produict ſon coſté.

CONSTRVCTION.

Soit le nombre requis	1 ①	2
Qui multiplié par 8 donné, faict pour premier produict	8 ①	16
Et multiplié ledict nombre requis par 2 donné, faict pour ſecond produict	2 ①	4
Qui doibt eſtre la racine du premier produict, ergo ſon quarré	4 ②	16
Eſt egal au ſecond en l'ordre	8 ①	16

Leſquels reduicts, 4 ① ſeront egales à 8, & par le 67 probleme 1 ① vaudra 2.

Ie di que 2 eſt le nombre requis. *Demonſtration.* Multiplié 2 par 8, faict 16, & le meſme 2 multiplié par le 2 donné faict 4, qui eſt racine quarrée, du quarré 16 ſelon le requis; ce qu'il falloit demonſtrer.

Figure 2. Stevin constructs a number. Arithmetique (1585), p. 465, livre I, question XXIX of Diophantus' Arithmetic.

The right margin of the construction makes explicit the running total produced by the series of procedures. Stevin used print formatting to show the progressive construction of the sought value whose veracity could then by demonstrated by plugging it back into the statement of the question.[32]

The French humanist Guillaume Gosselin, a well-connected courtier whose interests ranged from scholarship to pedagogy, gave "arithmetical demonstrations" for the rules he offered in his 1577 *Books on the great art or on the hidden region of numbers, which is vulgarly called algebra and almucabala*.[33] The term would catch on: in his 1657 *Mathesis Universalis* Wallis would use it in a theoretical discussion of arithmetical, geometrical, and algebraic ways to demonstrate books II and V of Euclid's *Elements*.[34] Using a term pioneered in Petrus Ramus'

[32] For the effects of reading Diophantus on the arithmetical style of another early modern reader, the algebraist Rafael Bombelli, see Morse (1981), 88 ff.

[33] Guillaume Gosselin, *De arte magna…libri quattuor* (Paris: Gilles Beys, 1577), fol. 57v, 59v, 61v = Gosselin (2016), 346 ff. See also Cifoletti (1995), 1409. On Gosselin's patchy biography, see Gosselin (2016), 49-58.

[34] John Wallis, *Opera Mathematica*, 3 vol. (Oxford: Sheldon Theater, 1693-9), vol. I, 118-126 and 183-193.

Algebra that (as we saw) Xylander used to generalize Diophantus' results, Gosselin called his rules "canons."[35] Like Cardano's rules, Gosselin's canons are rules applicable to particular forms of the equation. Again as with Cardano's rules, the form identifies values that are used in calculating the root. In the second canon, a square is equal to some sides and a number ($x^2 = bx + c$). Take half the number of sides and square them. Add the number. To the side for which this new number is the square (or in modern terminology, to the square root of this number), add half the number of sides: now we have $b/_2 + \sqrt{b^2/_4 + c}$, an equation that should remind readers of the quadratic formula for $a = 1$ (the difference in sign follows Gosselin's way of writing the equation). This is equal to x, the value of the side. The canon is general; the form it depends on, here "1 Q is equal to sides and numbers," marks out the limits of the procedure's generality.[36]

But the arithmetical demonstrations are particular, and through them, canons carried out on particular numbers serve to show their own generality. Gosselin adduced the example "6 L P. 16 equals 1 Q": six lengths plus sixteen is a square.[37] In order to solve the example, Gosselin employed his canon, justifying each step of its application.

> We divide 6 plus 2 A into A and 6 plus A, half of 6 plus 2 A is 3 plus A, therefore following of the fifth theorem of the second [book] of Euclid, [the product] made out of A and 6 plus A with the square of the excess [of] 6 plus A over 3 plus A, is equal to the square of half of the whole thing — that is, 3 plus A, and [the product] made out of A and 6 Plus A is 16, as is having to be demonstrated.[38]

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[35] Petrus Ramus also used the term "canon" in his *Algebra* ([Paris: Andre Wechel, 1560], fol. 13v, 15v, 16v). The very close similarity between Ramus' use and Gosselin's (both use canons to differentiate between what Cardano called "modes" of the equation, and they are the same three modes) suggests that Ramus was Gosselin's source. On Ramus' canons, see François Loget, "De l'algèbre comme art à l'algèbre pour l'enseignement: Les manuels de Pierre de La Ramée, Bernard Salignac et Lazare Schöner," *Revue de synthèse* series 6, 132.4 (2011): 495-527, at 500.

[36] Gosselin (1577), fol. 59r = Gosselin (2016), 351.

[37] Gosselin (1577), fol. 59v = Gosselin (2016), 351.

[38] Gosselin (1577), fol. 60r = Gosselin (2016), 351-3. Le Guillou-Kouteynikoff argues on the contrary for the generality of Gosselin's demonstrations on the grounds that the demonstration could be repeated for different numbers: see Gosselin (2016), 127. But note that whether the calculations can be performed "in the same way" on other numbers is the very question at stake in the difference between arithmetic and geometry.

Gosselin's arithmetical demonstrations could appeal to explicit axioms no less than to Euclid.[39] But these were not demonstrations whose validity rested on formal generality, for the particularity of the worked-out canon did not impede the demonstration from being general.[40] Arithmetical writers evidently believed that demonstrations based on particular cases were sufficient to produce belief.[41]

Visual and noetic means of understanding

During the sixteenth century algorithmic and geometric constructions resembled each other. On one hand, interest in both practical mathematics and mathematical pedagogy supported an explicit materialization of geometric construction. The materiality of constructions was reflected in the rising importance of mathematical instruments.[42] On the other hand, formalizing within arithmetic the use of Indo-Arabic numerals and other signs, as well as written procedures for their manipulation, involved making algorithmic constructions as visible as geometric ones. This section identifies two kinds of rhetorically-inflected algorithmic demonstrations and distinguishes the different techniques of persuasion that they use.

Early modern education reformers in general sought new ways visually to organize and present classroom material in order to make it easier to navigate and learn, and mathematics was

[39] On Gosselin's appeals to axioms, see Cifoletti (1995), 1409.
[40] Questions of formality in geometry are complicated by the role of "formulas" in Greek expression: see Reviel Netz, *The Shaping of Deduction in Greek Mathematics* (Cambridge: Cambridge University Press, 1999), ch. 4.
[41] On the related epistemic state of "assent" as a goal of geometrical demonstration, see Abram Kaplan, "Analysis and Demonstration: Wallis and Newton on Mathematical Presentation," *Notes and Records of the Royal Society* 74.2 (2018): 447-468.
[42] On instrumentation, see, *inter alia*, Alexander Marr, ed., *The Worlds of Oronce Fine: Mathematics, Instruments, and Print in Renaissance France* (Donington: Shaun Tyas, 2009) and discussion of the organic construction of curves in Niccolò Guicciardini, *Isaac Newton on Mathematical Certainty and Method* (Cambridge, MA: MIT Press, 2009). A general discussion of the "organic" approach in early modern geometry remains a desideratum.

understood as a model in this regard.[43] Mathematics teachers emphasized the necessity of visual organization of various kinds for promoting understanding. Pierre Hérigone, author of a multivolume *Cursus Mathematicus* or "course in mathematics" printed double-column in both French and Latin, claimed to introduce a system of "notes" or signs into Euclid's *Elements* that would "without using any language" present each further statement "explained and allowed from the premises" that have come before.[44] For the English author William Oughtred, one of the great advantages of symbolic algebra was its ability to make arithmetical operations visible: symbols neither rack the memory "with multiplicity of words, nor chargeth the phantasie with comparing and laying things together; but plainly presenteth to the eye the whole course and processe of every operation and argumentation."[45] Making these processes transparent would continue to occupy certain mathematical writers later in the seventeenth century.[46]

Attempts to visualize mathematics drew on a tradition of Ciceronian rhetoric that figured centrally in sixteenth-century algebra. As late as 1640, Hérigone repeated a commonplace argument of the earlier rhetorical-mathematical tradition: he claimed that his new notes managed the difficult but necessary goal of joining "brevity and clarity" in a single pedagogical method.[47] In the decades around 1550, the algebraist and poet Jacques Peletier drew from Ciceronian rhetoric key concepts that served to organize algebra as a new kind of discourse, notably "invention" and

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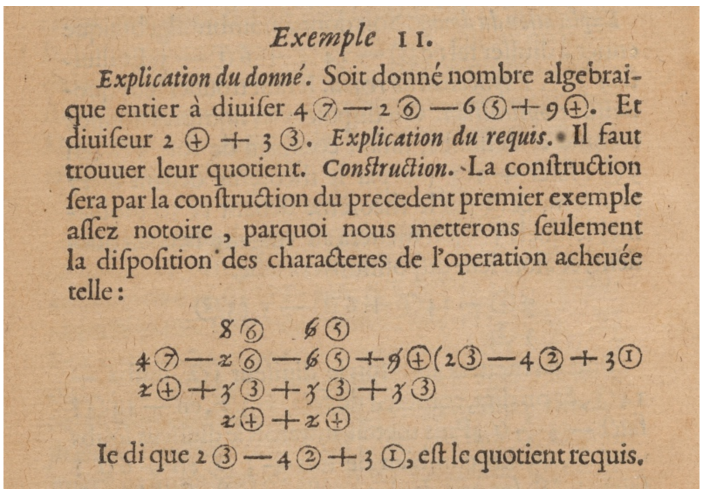
[43] Oosterhoff (2018); on the importance of math for Ramus, see Robert Goulding, *Defending Hypatia: Ramus, Savile, and the Renaissance Rediscovery of Mathematical History* (Dordrecht: Springer, 2010).
[44] Pierre Hérigone, *Cursus Mathematicus, nova, brevi et clara methodo demonstratus… Tomus Primus* (Paris: Henry Le Gras, 1634), Ad lectorem/Au lecteur (not paginated). On Hérigone, see also Maria Rosa Massa Esteve, "The Roel of Symbolic Language in the Transformation of Mathematics," *Philosophica* 82 (2012): 153-193.
[45] William Oughtred, *Key to the Mathematics*, trans. Robert Wood (London: Thomas Harper for Richard Whitaker, 1647), B4i. For discussion of this quotation, see Helena Pycior, *Symbols, Impossible Numbers, and Geometric Entanglements* (Cambridge: Cambridge University Press, 1997), 114.
[46] See e.g. John Pell's three-column tables for organizing algebraic manipulations: Noel Malcolm and Jacqueline Stedall, *John Pell (1611-1685) and His Correspondence with Sir Charles Cavendish: The Mental World of an Early Modern Mathematician* (Oxford: Oxford University Press, 2004), 268 ff.
[47] See the title and "Ad lectorem" to Hérigone (1634). Hérigone's text was printed in double-column in both Latin and French and the two are sometimes lightly different. Ad lectorem. On brevity and clarity, see Cifoletti (1992), *passim*.

"disposition."[48] For Peletier algebra served as an alternative to traditional modes of geometric presentation, one that involved only the "points necessary to resolve a difficulty."[49] A related attempt to use algebra to streamline geometrical argument is evident in the written constructions through which Stevin incorporated algorithms into the Euclidean demonstration scheme. To wit, he described as a "disposition" of numbers a long-division algorithm that could be used to divide polynomials (fig. 3).


Exemple II.

Explication du donné. Soit donné nombre algebraique entier à diuiſer 4⑦ — 2⑥ — 6⑤ + 9④. Et diuiſeur 2④ + 3③. ***Explication du requis.*** Il faut trouuer leur quotient. ***Conſtruction.*** La conſtruction ſera par la conſtruction du precedent premier exemple aſſez notoire, parquoi nous metterons ſeulement la diſpoſition des characteres de l'operation acheuée telle:

8⑥ 6⑤
4⑦ — 2⑥ — 6⑤ + 9④ (2③ — 4② + 3①
2④ + 3③ + 3③ + 3③
2④ + 2④

Ie di que 2③ — 4② + 3①, eſt le quotient requis.

Figure 3. Long-division algorithm for polynomials. Stevin, Arithmetique *(1585)*, book II, prob. X, example II, p. 234.

First given numbers were disposed; then the parts of the quotient could be disposed, following theorems given earlier in the text, until the final quotient was reached.[50] For Stevin, disposition was the algorithmic practice that, in parallel to the construction of a geometric figure in a traditional geometric demonstration, manifested the sought quantity.

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[48] Cifoletti (1995), 1389-1397. Judgment was another key concept imported into mathematical thinking from the rhetorical tradition, in particular by Ramus.

[49] Jacques Peletier, *L'algebre* (Lyon: Jean de Tournes, 1554), 1; on Peletier's understanding of the relationship between Euclidean demonstration and syllogism, see Giovanna Cifoletti, "From Valla to Viète: The Rhetorical Reform of Logic and Its Use in Early Modern Algebra," *Early Science and Medicine* 11.4 (2006): 390-423, at 403-6.

[50] Stevin (1585), 169, 232. On Stevin and the rhetorical tradition, see also Jean-Marie Coquard, "Mathématiques et dialectique dans l'oeuvre de Simon Stevin: l'intérêt des series de problèmes," *SHS Web of Conferences* 22.00012 (2015).

Because they are not visualized, meanwhile, Gosselin's canons instance another way in which algorithmic practice gave rise to understanding. Disposition was not unknown to Gosselin: elsewhere in his *Great Art* he showed readers how to set up an array in order to keep track of the many interim quantities produced by the rule of double false hypothesis.[51] But the arithmetical demonstrations of Gosselin's canons were supposed to produce understanding—or at least the lesser goal of producing belief—without explicit reliance on visuality or other rhetorical resources. And on its face this seems like a problem for arithmetical demonstration in general, since particular numbers evidently lack the indeterminacy of magnitude that makes induction from particular geometric constructions to general geometrical theorems seem plausible to us.[52] Sixteenth-century algebraists understood their demonstrations to depend on rhetorical *loci communes*, commonplaces that they associated with internalized common notions.[53] This observation may help explain why the purpose of demonstrations was to produce belief, a classic goal of classical rhetoric. But it leaves unspecified how particular instantiations of more general axioms served to evidence general canons. Looking at Gosselin's particulars sheds light on how, within the framework of algebraic demonstrations' dependence on common notions, algorithms could produce belief. *A posteriori*, Gosselin's canons also shed light on how appeal to common notions worked.

Gosselin sandwiched the "demonstration" of each canon between two other parts. The first part introduced the rule, while the third provided the "use" of the rule for the resolution of a "problem."[54] All three parts involve particular, always different examples. The examples in the demonstration are consistently of medium difficulty between the extremely simple example in the

[51] See Gosselin (1577), fol. 23r-24v = Gosselin (2016), 277-81.
[52] I cannot pursue this question here. But for near contemporary recognition of the problem see Wallis (1695), 53.
[53] On the role of commonplaces as "proofs" in algebra, see Cifoletti (1995), 1391.
[54] See Gosselin (1577), fol. 57v ff. = Gosselin (2016), 347 ff.

statement and the difficult example in the problem. For Gosselin, "uses" of the rule frequently involve fractions and three-digit numbers.[55] "Demonstrations" never do: here Gosselin prefers relatively small whole numbers, as in the example discussed above: "six lengths plus sixteen is a square." Meanwhile, the examples in the statement of the rule are even smaller: "A square equals four times its side plus twelve."[56]

Since the example of median difficulty is neither more nor less true than any other example, its suitability for the demonstration must come from elsewhere. Given Gosselin's consistency in choosing the difficulty of his examples, it seems that Gosselin tailored the difficulty of each to its purpose within his presentation. The very simple example given in the statement allows the rule to be immediately grasped as plausible. The most difficult example given in the form of a problem allowed Gosselin to show off the application of the rule in a case that exceeds unaided human powers of numerical reckoning, at least for most people. Meanwhile, I suggest, the example of median difficulty renders the demonstration more persuasive than either alternative would.

Using the middle example, the right reader can verify the procedure by following its steps in the mind. The level of difficulty of the median examples is low enough that the reader can verify each step rather quickly. Meanwhile, the difficulty is not so low that an unpracticed reader can see the answer immediately without working through the procedure. Such examples could be verified in the course of the demonstration, since the reader could compare his or her results against the intermediate sums and products Gosselin provided. The examples are thus persuasive in the sense

[55] Cf. Xylander's edition, where both the Greek scholiast Maximus Planudes and Xylander himself introduce fractional examples for problems that Diophantus broaches in whole numbers. At one point Planudes explains the practice: "Idque exercitationis causa plenioris demonstremus in numeris fractis" (Diophantus [1575], 35). The juxtaposition between "we demonstrate" and "for the sake of fuller exercise" suggests that "demonstration" does not have the philosophical significance often attributed to the word. On this significance, see discussion of the "question of the certitude of mathematics" in Jardine (1988) and Mancosu (1992).
[56] Gosselin (1577), fol. 59v = Gosselin (2016), 351.

that their truth is made available to the reader in an inward way, through the intellectual use of the procedure.

The notion of inward persuasion is consistent with a broad Platonist trend in the epistemology of mathematics adduced in sixteenth-century French arithmetic and algebra books. A vague but vividly illustrated Platonist epistemology often coexisted, in the same textbook, with efforts to ameliorate mathematical pedagogy through new formatting and rhetorical formalism and with correlated appeals to rhetorical epistemology. This vivid Platonism emphasized the interiority of understanding rather than the visualization of knowledge. Textbook authors drew on standard Platonic *topoi*, such as the image of ascent from the *Phaedus* and the image of conversion from the *Republic*, to frame mathematical learning as the best way that students could first experience intellectual transformation. Some curricular reformers thus assigned mathematics a central role in liberal education.[57] For certain mathematicians, meanwhile, mathematical learning cultivated souls and gave direct, experiential knowledge of metaphysical principles.[58] In a somewhat hyperbolic adaptation of Plato's allegory of the cave, Peletier claimed that "there is no speculation that can serve man with a more spacious country to frolic, to explore his thoughts, to draw him outside of himself and then assume himself again, than the university of numbers."[59] Experience with mathematical things led, he believed, to self-knowledge and understanding.

The organization of Gosselin's canons, clearly, does not advance so comprehensive a view of the relationship between mathematics and philosophy. But Gosselin's use of examples reflected his own understanding that mathematical education aimed to cultivate inward understanding. The

[57] Boethius' *De institutione arithmetica* was often central to this presentation. See Moyer (2012); Oosterhoff (2018).
[58] For one example, see discussion of Peletier in Natalie Zemon Davis, "Peletier and Beza Part Company," *Studies in the Renaissance* 11 (1964): 188-222. For Peletier's philosophy more generally, see Sophie Arnaud, *La voix de la nature dans l'œuvre de Jacques Peletier du Mans (1517-1582)* (Paris: Honoré Champion, 2005).
[59] Peletier (1554), 122-123.

persuasiveness of Gosselin's arithmetical demonstrations depended on this previously cultivated understanding. A contrary emphasis on the generality of the mathematical object in Gosselin's *Renewed lecture* reflects the absence of such an understanding: that text is pitched at beginning arts students without meaningful familiarity with mathematics.[60]

Gosselin's commitment to the mind contrasted with the emphasis on visualization evident elsewhere in the algebraic tradition. Hérigone's method of notes, for instance, did not serve just to visualize the steps of a demonstration but to visualize *all* the steps. "There is no doubt that understanding is easier than with the common way of demonstrating," he wrote, "since in my method nothing is asserted unless it is corroborated by some citation. Other authors do not exactly observe this, but, measuring the necessity of citations from those things which seem open or obscure to them, they use many consequences without any citations, which nevertheless everyone knows are of great help to uneducated and less exercised [readers]."[61] Where Peletier and others turned to rhetorical algebra in order to cut away the fat from geometric demonstration, Hérigone wanted to bulk up the demonstrations that he expressed using symbols. Only such an exhaustive approach would make it easy "to convert" or "to resolve into syllogisms" each line of a given demonstration.[62] For Hérigone and other early modern textbook writers, making it explicit—through notes or other formalism—served pedagogy, no less than new modes of writing like Stevin's dispositions served effective practice. Both could serve as aids to those who lacked understanding, since canons and rules can be used without understanding their reasons, as they still are today by students of high-school algebra. But neither sort of writing furnished an alternative kind of knowledge than that which implicated the self. Demonstrating Gosselin's

[60] See Gosselin (2016), 417-9 = fol. 5r/v.
[61] Hérigone (1634), Ad lectorem.
[62] Hérigone (1634), "Ad lectorem."

canons depended on a firm grasp not just of common notions but also of how to use them to calculate. In the context of the *Great Art* this included experience with, and knowledge of, the behavior of numbers as well.

Understanding algebra

By way of conclusion it will profit us to look briefly at the mathematics of François Viète. Like his contemporary Gosselin, Viète was inspired by the recovery of Diophantus' *Arithmetic*; unlike Gosselin, he took Diophantus' study of higher powers of the unknown in a mostly new direction.[63] Viète set out to understand indefinite equations, which he understood as proportions between "species" of quantities (the same sides, squares, and cubes that we have already seen) and which he represented using a combination of abbreviations, short words, and symbols reminiscent of Diophantus' own text. Viète's interest in and success at studying equations has led many scholars to see Viète's mathematics as a major step in the direction of contemporary mathematics. And this is true regardless of how they interpret Viète's algebra, which usually means what ontology they attribute to his equations: whether they think that Viète's algebra identifies some new object common to arithmetic and geometry or that it, rather, is an object-independent mathematical tool along the lines of the *trivium*.[64] Regardless of the interpretation, however, Viète's symbolic mathematics can hardly be depicted as purely formal and automatic. For it involves all three cognitive practices we have seen: form as an indicator of generality and its limits,

[63] For Arabic precedents for Viète's mathematics, see Roshdi Rashed, *The Development of Arabic Mathmetics: Between Arithmetic and Algebra*, trans. A. F. W. Armstrong (Dordrecht, Boston, and London: Kluwer Academic Publishers, 1994).
[64] See Klein (1992); Michael Mahoney, "The Beginnings of Algebraic Thought in the Seventeenth Century," *Descartes: Philosophy, Mathematics and Physics*, ed. Stephen Gaukroger (Sussex: Harvester Press, 1980), 141-168; Marco Panza, "What is New and What is Old in Viète's *Analysis Restituta* and *Algebra Nova*, and Where Do They Come From?" *Revue d'histoire des mathématiques* 13 (2007): 85-153; and, for discussion of these alternatives, Rabouin (2009).

algorithm as a means of demonstration, and a dependence on inward understanding for justification.

Viète followed Diophantus in distinguishing multiple "zetetics," finding aids, based on different givens.[65] Like Cardano's rules and Gosselin's canons, Viète's zetetics had their own domains of applicability. Once the right zetetic was identified, it could be used automatically to find the solution to a given problem. But Viète's justifications were not automatic. Seventeenth-century algebraists would resolve their equations through piecemeal manipulation, raising and lowering by degree until they arrived at a solution.[66] Viète's *Introduction to the Analytic Art* set out rules for doing just this.[67] But Viète rarely used such a piecemeal approach in his demonstrations. Instead, the demonstrations employed relationships between different problems by referring the reader back to earlier zetetics. This reference practice instanced Viète's reliance on inward understanding for justification. Unlike Hérigone, he didn't spell out in great detail how to get a new demonstration from an earlier one; instead, he referred generally to operations that he expected his reader to know how to use.[68] Treatises on more advanced topics were pitched at a higher level, and Viète's use of earlier finding aids in new demonstrations was even less explicit.

Finally, with Viète we get a near-total fusion of algorithms and demonstrations, since the finding aids are at once part of the discourse and the practice. Like the relationship between a Euclidean figure and its text, Stevin's dispositions were printed in parallel to the language that explained them. If one were actually to divide an equation using Stevin's method, the justificatory

[65] On parallels between Viète's zetetica and Diophantus' *Arithmetic*, see Paolo Freguglia, "Viète Reader of Diophantus. An Analysis of *Zeteticorum Libri Quinque*." *Bollettino di storia delle scienze matematiche* 28.1 (2008): 51-95.
[66] For example, Pell's three-column method mentioned earlier keeps track of just this kind of manipulation. See Malcolm and Stedall (2004).
[67] François Viète, *Opera Mathematica*, ed. Franz van Schooten (Leiden: Elsevier, 1646), 1 ff.; and in English translation, see François Viète, *The Analytic Art*, trans. T. Richard Witmer (Kent, Ohio: Kent State University Press, 1983), 11 ff.
[68] See, e.g., *Zeteticorum libri quinque*, lib. II, zetetica 3-8, in Viète (1646), 51-2.

language would be unnecessary; one could simply take recourse to the operative disposition. Viète exempted himself from using such a language. Although the parallel to Diophantus' *Arithmetic* might suggest it, Viète's freedom from the double-discourse was not simply a matter of symbols and abbreviations. It also derived from Viète's willingness to rely on the reader's powers of recognition even when he silently used earlier finding aids to transform one formal expression into another. The persuasive aspect of Viète's symbolic discourse depended on the recognition of equations, on the interaction between forms as markers and inward understanding.

To return to our orienting question: does the use of algorithms require humans to behave like machines? Of course, answering this depends on how (and whether) machines behave. But what's clear is that sixteenth-century algorithms offered many possibilities for use and understanding, from the automatic to the apprehending. I can make apposite comparison to two other chapters in this volume. Michael Barany argues for the importance of experience and discernment to the correct use of early modern algorithms. In the examples we've seen, discernment is key to recognizing which algorithm to use, and experience plays a central role in understanding demonstrations and so reasons. But the kind of algorithmic facility with the rule of three that Caitlin Rosenthal finds in the Atlantic business life circa 1800 was also one possible outcome of an encounter with sixteenth century algebras.

Nor is this diversity limited to algorithms. In their seminal work on humanist education, Anthony Grafton and Lisa Jardine identified the emergence of the "humanities" with Ramus' decision to shift the emphasis of his pedagogy from cultivating the soul to inculcating practices.[69] As though different outcomes and different kinds of understanding were not always possible even within a single classroom, let alone from a whole pedagogical movement. The history of how

[69] Grafton and Jardine (1986).

algorithms have been understood should remind us that when it comes to understanding, there are degrees.